NUMERICAL MODELING OF SUBGRID-SCALE FLOW IN TURBULENCE, ROTATION, AND CONVECTION

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We show that it is impossible to simulate numerically all of the length-scales in astrophysical turbulence. We look at the effects of ignoring the unresolvable, small-scale flow, and show how numerical simulations that neglect the subgrid-scale motions produce erroneous solutions. Then we discuss the "quick fix" remedy of introducing a numerical or eddy-viscosity. We sketch how the analytic theories of turbulence attempt to model the large-scale effects of the small-scale motions. In the last section of this paper we examine four anisotropic, inhomogeneous flows of astrophysical interest for which numerical and eddy-viscosities produce incorrect solutions. Improved models of the subgrid-scale flow are examined. We show how numerical simulation of large Reynolds number (but non-turbulent) flows can guide us in modeling subgrid-scale flows in astrophysical settings.

Turbulence, rotation, and convection are generally treated by astrophysicists in the same manner; they are avoided whenever possible. Jets do not become turbulent, stars do not rotate, and planetary nebulae do not convect - except when the absence of these motions contradicts common sense or when the mixing properties of these motions must be invoked to solve some astrophysical paradox. The reason we avoid calculating these flows is that the numerical computation of turbulence, in even the simplest laboratory settings, is usually impossible because the velocity spans a much larger range of length-scales and has many more degrees of freedom than can be accommodated in present computers. The

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calculation of turbulent flows therefore requires us to model or neglect a large portion of the flow. In this paper, we shall not be concerned with the technical details of computing derivatives or matching boundary conditions but with the more fundamental problem of how the equations of motion should be changed so that the flow at the largest scale can be accurately simulated while the large-scale effects of the numerically unresolvable or subgrid-scale motions are modeled. Our task is analogous to the one of finding a simple spatial boundary condition: it is impractical to compute unbounded flows (for example, a stellar wind) so at some finite distance far from the physics of interest, the calculation is cut off and an artificial but physically appropriate boundary condition is imposed (for example, all characteristics must point outward at the boundary). With turbulent flows it is impractical to compute the flow at all spatial scales or equivalently at every point in Fourier space, so at some large, but finite, wavenumber far away from the physics of interest the calculation is cut off and an artificial boundary condition is imposed that represents the effects of the subgrid-scales. As with all artificially imposed boundary conditions, the less information that propagates from the neglected region of the flow into the computational domain, the better the approximation. Boundary conditions in wave number space are more complicated than those in physical space because the equations of motion in wave space are non-local, allowing the subgrid-scale motions to affect directly the flow everywhere, not just near the boundary.

The philosophy of this paper is that at large scales, astrophysical flows do not exhibit much universality. Large-scale coherent features, such as the solar granulation or the Red Spot of Jupiter, cannot be predicted by an analytic theory of turbulence. They must be simulated numerically for every particular flow, while the small-scale structures in astrophysical flows may be universal. It is a fundamental assumption of this paper that, as energy cascades down from the large energy-producing scales to the small scales, the flow loses information about the large-scales such that the effects of the small scales can be modeled by a universal theory of turbulence.

FORCING DUE TO SUBGRID-SCALE MOTIONS

In most astrophysical flows, we are interested only in the large-scale velocity which is not affected directly by viscosity, so we use the inviscid Euler equation rather than the Navier-Stokes equation. The danger of ignoring viscosity is that it strongly influences the subgrid-scale flow which in turn acts upon the large-scale flow. To see the consequences of the numerically unresolved motions, it is necessary to realize that a numerical

simulation implicitly filters the velocity, $\underline{v}$, into a large-scale numerically resolvable component, $\overline{\underline{v}}$, and a subgrid-scale component, $\underline{v}'$. Mathematically, the numerical representation of the velocity convolves $\underline{v}$ with a filtering function $F(x,x')$:

$$\overline{\underline{v}}(x) = \int F(x,x')\underline{v}(x')d^3x' \tag{1}$$

$$\underline{v}'(x) = \underline{v}(x) - \overline{\underline{v}}(x) \tag{2}$$

The filtering function contains a characteristic length-scale, Δ, (e.g., the grid spacing in finite-difference methods) so that $\overline{\underline{v}}$ represents motions on length-scales larger than Δ and motions smaller than Δ are removed (and don't seriously contaminate $\overline{\underline{v}}$). $F(x,x')$ does not have to be homogeneous or isotropic, and although it is usually an implicit property of the finite-difference, spectral, or finite element method, it can also be explicitly selected by the numericist (cf. Moin, Reynolds, and Ferziger, 1978). In finite-difference calculations, it is often difficult to express $F(x,x')$ in closed form, but for a pseudo-spectral representation of a 1-dimensional periodic, real velocity field with wavelength 2π and with collocation points $x_p \equiv 2\pi\, p/N$, $p = 0,1,2,\ldots,N-1$, the convolution function $F(x,x')$ is easily determined:

$$\overline{\overline{v(x)}} = \sum_{k=-N/2}^{N/2} e^{ikx}\, \overline{v}(k) \tag{3}$$

where $\overline{v}(k)$ is the discrete Fourier transform

$$\overline{v}(k) = \frac{1}{N} \sum_{p=0}^{N-1} e^{ikx_p}\, v(x_p) \tag{4}$$

so that

$$\overline{\overline{v(x)}} = \sum_{p=0}^{N-1} v(x_p)\, \frac{\cos[\frac{N}{2}(x-x_p)]-\cos[(\frac{N}{2}+1)(x-x_p)]}{N[1-\cos(x-x_p)]} \tag{5}$$

Therefore,

$$F(x,x') = \sum_{p=0}^{N-1} \delta(x'-x_p)\, \frac{\cos[\frac{N}{2}(x-x')]-\cos[(\frac{N}{2}+1)(x-x')]}{N[1-\cos(x-x')]} \tag{6}$$

In equation (6), the characteristic length of the filter is $\Delta = 2\pi/N$.

The pressure, temperature, density, and all other quantities of the flow are similarly filtered. Nonlinear terms in the

equations of motion couple the subgrid flow to the large-scale flow. The numerically filtered Navier-Stokes looks like the Euler equation written in terms of the large-scale pressure, P, density, ρ , and velocity:

$$\partial \underline{\bar{v}}/\partial t = \overline{-(\underline{v}\cdot\nabla)\underline{v}} - \overline{\nabla p/\rho} + \overline{1/\rho\ \nabla\cdot\underline{\underline{\Pi}}} + \underline{Q} \qquad (7)$$

where $\underline{\underline{\bar{\Pi}}}$ is the large-scale component of the viscous stress tensor, and $\underline{Q}$ contains all terms through which the subgrid-scales affect the large-scale flow:

$$\underline{Q} = \overline{-(\underline{v}'\cdot\nabla)\underline{\bar{v}}} \quad \overline{-(\underline{\bar{v}}\cdot\nabla)\underline{v}'} \quad \overline{-(\underline{v}'\cdot\nabla)\underline{v}'} + \overline{(\rho'/\bar{\rho})[1/(\bar{\rho}+\rho')]\nabla(\bar{P}+P')} \quad \overline{-\nabla P'/\bar{\rho}} - \overline{(\rho'/\bar{\rho})[1/(\bar{\rho}+\rho')]\ \nabla\cdot(\underline{\underline{\bar{\Pi}}}+\underline{\underline{\Pi}}')} \quad \overline{-\nabla\cdot\underline{\underline{\Pi}}'/\bar{\rho}} \qquad (8)$$

For large Reynolds number flows, the viscous contributions to equations (7) and (8) are negligible and will henceforth be ignored, but the subgrid-scale term, $\underline{Q}$, in equation (7) is large and important. Viscosity indirectly influences the value of $\underline{Q}$ by changing $\underline{v}'$. The main danger of neglecting $\underline{Q}$ in numerical simulations is that the computed flow may be numerically stable and exhibit many interesting astrophysical properties, but not look at all like the true stable equilibrium.

IMPOSSIBILITY OF RESOLVING THE SMALL SCALES

To understand our present inability to simulate motion at all scales and the need of modeling the subgrid-forcing, $\underline{Q}$, we need to review some aspects of turbulence theory. A turbulent fluid is made up of motion spanning a wide range of length-scales, ℓ , each scale with its own characteristic velocity $v(\ell)$. In all theories of turbulence (with a small but non-zero viscosity), $v(\ell)$ decreases with decreasing ℓ so there is a characteristic length ℓ_M such that the velocity of scales with $\ell < \ell_M$ have sufficiently small Mach number so, although the large-scale flow is compressible, the small-scale flow is incompressible. If we assume that we have sufficient numerical resolution so that all supersonic and transonic scales are resolved, i.e., $\ell_M > \Delta$, then the subgrid-scale flow is incompressible and we can confine our discussion of turbulence to incompressible flow.

Consider a high Reynolds number flow driven by some external force (e.g., buoyancy) at some length-scale ℓ_b such that the rate of kinetic energy per unit mass per unit time entering the flow is ε. According to the classic Kolmogorov theory of turbulence (1941), there is a net cascade of energy downwards from ℓ_b to a small dissipative length-scale, ℓ_d. In the Kolmogorov picture, the cascade is non-dissipative so that the rate at which energy crosses from motions with scales greater than ℓ into motions less than ℓ is ε, where the rate is approximated as

$$\varepsilon \simeq v(\ell)^3/\ell \quad \text{for} \quad \ell_b \geq \ell \geq \ell_d \tag{9}$$

Equation (9) is the source of Kolmogorov's famous scaling law, $v(\ell) \propto \ell^{1/3}$. For an incompressible fluid, the rate at which viscosity dissipates energy at length-scale ℓ is $\nu\, v(\ell^2)/\ell^2$ where ν is the kinematic viscosity; therefore, since $v(\ell) \propto \ell^{1/3}$, viscous dissipation is most effective at small length-scales. By assuming that all of the kinetic energy is dissipated at the smallest length-scale ℓ_d, we obtain an estimate for ℓ_d.

$$v(\ell_b)^3/\ell_b = \varepsilon = \nu v(\ell_d)^2/\ell_d^2 \tag{10}$$

$$= \nu[v(\ell_b)(\ell_d/\ell_b)^{1/3}]^2/\ell_d^2 \tag{11}$$

or

$$\ell_b/\ell_d = R^{3/4} \tag{12}$$

where R is the Reynolds number based on the largest length-scale

$$R = v(\ell_b)\ell_b/\nu \tag{13}$$

[Equation (12) for determining ℓ_b/ℓ_d can be derived in an alternate manner by assuming $v(\ell) \propto \ell^{1/3}$ and finding the length-scale ℓ_d where the characteristic inertial term of the Navier-Stokes equation, $v(\ell_d)^2/\ell_d$, is equal to the viscous term, $\nu v(\ell_d)/\ell_d^2$.] Equation (12) contains all the information a numericist needs to compute his computer budget for a simulation. It says that to include all of the physically important length-scales from the largest, ℓ_b, to the smallest, ℓ_d, he needs at least $R^{3/4}$ finite-difference points (or spectral modes or finite elements) per spatial dimension. Typical astrophysical flows have large Reynolds numbers. Solar convection has a Reynolds number of 10^{14} which means that a full 3-dimensional simulation requires 10^{31} grid points! With today's supercomputers, calculations with greater than 10^6 grid points are not practical; therefore, in solar convection all scales less than $\Delta = \ell_b/100$ must be neglected or modeled. Although modern experiments and theories of turbulence

(see below) show that the Kolmogorov picture is too simple, the estimate in equation (12) for the dissipative length-scale and hence the estimate for the computer budget is still valid.

NEGLECT OF THE SUBGRID-SCALES

Neglecting the subgrid-scales deprives the flow of its natural outlet for dissipating kinetic energy. For simulations of flows with Reynolds numbers of a few hundred in which $\Delta > \ell_d$ the kinetic energy cascades down from ℓ_b to Δ, where it has no place to go and begins to pile up. It continues to accumulate at Δ until the velocity $v(\Delta)$ is sufficiently enhanced so that the rate of dissipation at Δ, $\nu v(\Delta)^2/\Delta^2$, balances ε. Figure 1 shows the kinetic energy spectrum of a pseudo-spectral simulation of Taylor-Couette flow between two cylinders at a Reynolds number of 460. The abscissa in figure 1 is the axial wavenumber, $k \simeq \ell_b/\ell$, and k=15 is the limit of the numerical resolution. In a well-resolved simulation (Marcus, 1983) the energy spectrum for k>3 is a straight line and the upward curl in the energy spectrum at k=15 in figure 1 is due to the fact that $\Delta > \ell_d$ in this calculation. Repeating the calculation with twice as much resolution makes $\Delta < \ell_d$ and the spectrum becomes a straight line from k=4 to k=31.

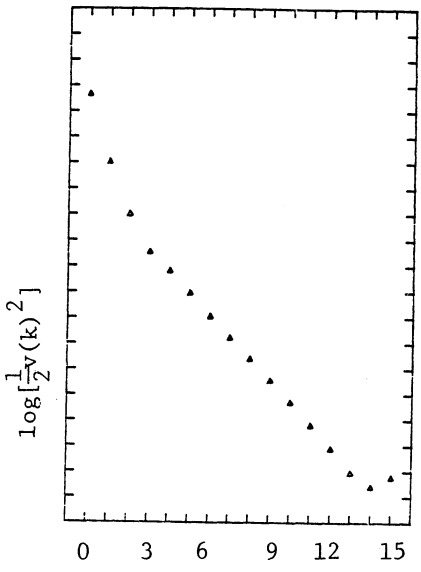

Figure 1 - Under-resolved energy spectrum

In most calculations at Reynolds numbers of a few hundred, insufficient numerical resolution manifests itself by producing artificial small structures with size approximately Δ . In some simulations, the lack of resolution produces other types of spurious results. For example, single-mode theory (Toomre, Gough, and Spiegel, 1977) is a numerical method in which all information in two of the three spatial dimensions is represented spectrally with a basis function (usually a linear combination of Fourier modes or some other eigenfunction of the 2-dimensional Laplacian) and information in the third spatial dimension is computed with a fine mesh of finite-difference grid points. Most astrophysical applications of single-mode theory have been with stellar convection (Toomre, Zahn, Latour, and Spiegel, 1976) in which the vertical direction is treated with finite-differences and the horizontal direction is represented with rolls or hexagonal planforms. The horizontal resolution is $\Delta_h = \ell_b$ and the vertical resolution is $\Delta_v << \ell_b$ with $\ell_v >> \ell_d$. Since the kinetic energy is prohibited by the numerics from cascading into small 3-dimensional motions, the flow forms artificially thin 1-dimensional boundary layers. The vertical thickness of the boundary layers, δ , is estimated by equating the rate of viscous dissipation integrated over the boundary layer to the energy input integrated over the entire volume of the fluid:

$$(\rho\nu v^2/\delta^2)\ell_b^2\delta = \rho\varepsilon\ell_b^3 \tag{14}$$

or

$$\delta/\ell_b = \nu v^2/\varepsilon\ell_b^2 \simeq \nu/v\ell_b = R^{-1} \tag{15}$$

The relationship in equation (15) has been confirmed numerically by Marcus (1981).

Under-resolved numerical simulations can have other signatures. A statistically steady state requires that the viscous dissipation balance energy production. If the flow, due to limited resolution, is incapable of dissipating energy efficiently, the numerically computed flow can respond by artificially lowering ε . However, in thermal convection there is a strong constraint preventing ε in the computed flow from deviating significantly from its correct value. The constraint is due to the fact that in thermal convection, the energy input rate per unit mass is proportional to the convective flux, F_c .

$$\varepsilon = F_c/\rho\Lambda \tag{16}$$

where Λ is the density scale-height. Equivalently,

$$\varepsilon = (F + k c_p \rho \cdot \frac{\partial \langle T \rangle}{\partial z})/\rho\Lambda \tag{17}$$

where $\partial\langle T\rangle/\partial z$ is the horizontally-averaged vertical temperature gradient, F is the total stellar flux, and k is the thermal diffusivity, ($k = 4acT^3/3\chi\rho$). In stellar convection, $\partial\langle T\rangle/\partial z$ is negative and less than the adiabatic gradient, $\partial\langle T\rangle/\partial z|_{ad}$ (since we assume that the star is not radiatively stable). If the convection is efficient (i.e., ν and k are small), then $\partial\langle T\rangle/\partial z$ cannot be much greater than $\partial\langle T\rangle/\partial z|_{ad}$ (since even a small super-adiabaticity makes the flow advectively unstable if ν and k are small). Since F is fixed and $\partial\langle T\rangle/\partial z$ must be nearly equal to its adiabatic value, the kinetic energy input rate is

$$\varepsilon \simeq (F + k c_p \rho \frac{\partial\langle T\rangle}{\partial z}\Big|_{ad})/\rho\Lambda \qquad (18)$$

and is computed correctly by nearly all codes; ε is insensitive to the resolution of the calculation.

In thermal convection, the entropy variance, $s(\ell)^2$, is analogous to the kinetic energy. The variance created at the large scale ℓ_b cascades via the nonlinear inertial terms in the equations of motion to a small thermal dissipation scale, ℓ_T. If $\ell_T < \Delta$, the entropy variance piles up at Δ or the numerically calculated flow must adjust itself so that the rate of entropy variance production, ε_s, is artificially decreased. Unlike ε, the numerical value of ε_s is not strongly constrained. The variance production rate at large scales is proportional to ($\partial\langle T\rangle/\partial z|_{ad} - \partial\langle T\rangle/\partial z$)

$$\varepsilon_s \simeq \frac{F c_v}{\rho T^2}\left(\frac{\partial T}{\partial z}\Big|_{ad} - \frac{\partial T}{\partial z}\right) \qquad (19)$$

Although efficient convection makes ($\partial\langle T\rangle/\partial z|_{ad} - \partial\langle T\rangle/\partial z$) small, a change in its value from 0.001 to 0.1 changes the value of ε_s by 100; therefore, ε_s is not strongly constrained. A numerical simulation with $\Delta > \ell_T$ produces not only a pile-up of entropy variance at Δ, but also an erroneously small value of ($\partial\langle T\rangle/\partial z|_{ad} - \partial\langle T\rangle/\partial z$) and thereby reduces the value of ε_s.

Another problem of under-resolved calculations is the computation of the correlation between the temperature and the vertical velocity:

$$C(z) \equiv \frac{\langle T(\underline{x})\ v_z(\underline{x})\rangle}{[\langle T(\underline{x})^2\rangle\langle v_z(\underline{x})^2\rangle]^{1/2}} \qquad (20)$$

where angle brackets denote horizontal averaging. In calculations with insufficient horizontal resolution an artificially high value of correlation is produced; in single-mode calculations the correlation is identically equal to unity. Convection experiments in air with Reynolds numbers of a few hundred have a correlation of about 0.6 (Deardorff and Willis, 1967). We have already seen that single-mode calculations are constrained to produce a nearly correct value of ε . The temperature flux, $\langle T(x)\, v_z(x)\rangle$ is also constrained to be nearly its correct value since ε is proportional to it. Therefore, since C(z) is over-estimated and $\langle T(x)\, v_z(x)\rangle$ is nearly correct, single-mode calculations must necessarily under estimate $\langle T(x)^2\rangle$ or $\langle v_z(x)^2\rangle$. It has been shown that single-mode calculations tend to under estimate $\langle T(x)^2\rangle^{1/2}$ by a factor of about 2 (Marcus, 1981).

In an under-resolved simulation, as the Reynolds number is increased, $v(\Delta)$ increases until it becomes the same order as $v(\ell_b)$. When this happens the energy jumps abruptly back from the small scales to the large scales and then slowly cascades back to the small scales in a periodic oscillation. This periodicity is shown in figure 2, which was calculated with an under-resolved

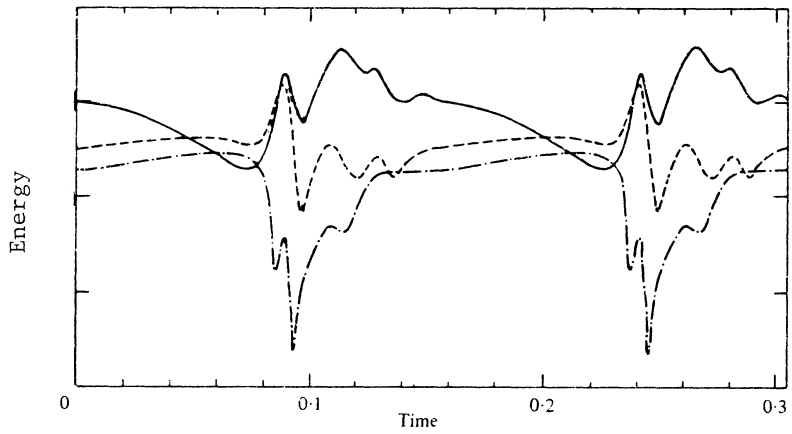

Figure 2

simulation of convection in a sphere. The energy as a function of time at length-scale ℓ_b is shown by the solid curve, at the smallest scale, Δ , by the dash-dot curve, and at an intermediate scale by the dashed curve. The time is in units of the thermal dissipation time. At t=0 the kinetic energy spectrum looks normal with energy decreasing with decreasing ℓ . As time advances, the energy piles up at Δ until the spectrum becomes inverted, with the largest scale containing the least amount of

energy. At t=0.08 the energy rushes back into the large scale and the oscillation repeats itself. The period of the oscillation is a function of ℓ_b/Δ. When the numerical resolution is doubled so that $\Delta < \ell_d$ and $\Delta < \ell_T$, the flow settles down to a steady state. Although ℓ_b/Δ controls the period of the oscillation, it is Δ/ℓ_d that determines whether there is a transition from a steady-state to a time-dependent one. Holding all physical quantities fixed, and increasing Δ/ℓ_d causes the flow to become periodic; if Δ/ℓ_d is increased further, the numerically computed flow becomes chaotic in time. All interesting temporal behavior in calculations with $\Delta/\ell_d >> 1$ or $\Delta/\ell_T >> 1$ is, of course, a numerical artifact.

SIMPLE EDDY-VISCOSITIES

There are several ways to prevent the pile-up of energy at small scales other than by increasing the resolution so $\Delta < \ell_d$. The easiest method is the introduction (accidental or planned) of numerical viscosity. The subgrid-scales affect the large scales by acting as conduits of kinetic energy from the large motions to the dissipative flow with an approximate rate of $v(\Delta)^3/\Delta$. In Kolmogorov's picture of turbulence, the cascade is local so the subgrid-scales drain energy mostly from the resolvable scales with approximate size Δ. Numerical viscosity can mimic this type of energy drain. Consider solving numerically Euler's equation with constant density by using upwind differencing. In one dimension this difference scheme is

$$\frac{\partial v_i}{\partial t} = -v_i \frac{v_i - v_{i-1}}{\Delta} - \frac{1}{2\rho}(P_{i+1} - P_{i-1}) \tag{21}$$

where $v_i = v(x_i)$, $x_i = i\Delta$, and we have assumed that $v_i > 0$. Equation (21) introduces a numerical viscosity which can be calculated by expanding v_{i-1}, P_{i+1} and P_{i-1} in a Taylor series about x_i. To second order in Δ, equation (21) is equivalent to

$$\left.\frac{\partial v}{\partial t}\right|_{x=x_i} = -v(x_i)\left.\frac{\partial v}{\partial x}\right|_{x=x_i} - \frac{1}{\rho}\nabla P\Big|_{x=x_i} + \left[\frac{1}{2}v(x_i)\Delta\right]\left.\frac{\partial^2 v}{\partial x^2}\right|_{x-x_i} + O(\Delta^2) \tag{22}$$

The numerical viscosity is $(\frac{1}{2}v(x)\,\Delta)$ and has several desirable properties: it is a positive-definite sink of kinetic energy, it draws its energy primarily from resolvable scales of size Δ, and

the rate of dissipation is approximately $v(\Delta)^3/\Delta$ (the last two properties are true if the flow has an energy spectrum such that $v(\ell)^3/\ell$ increases with decreasing ℓ and $v(\ell_b)$ and $v(\Delta)$ are poorly correlated - see below).

Although this crude numerical viscosity drains energy out of the small-scale motions, it usually dissipates a significant amount from the large-scale modes as well. To determine which scales the numerical viscosity dissipates, observe that the rate of energy dissipation at scale ℓ is proportional to

$$\frac{1}{2}\Delta \sum_{\ell'\ell''} C(\ell,\ell',\ell'')\ v(\ell)v(\ell')v(\ell'')/\ell'^2 \tag{23}$$

where $C(\ell\ \ell'\ \ell'')$ is a measure of the correlation among motions of scale ℓ, ℓ', and ℓ''. The velocity, $v(\ell)$, is made up of a band of Fourier modes with wavevectors k where $|k| \simeq \ell_b/\ell$. The correlation is proportional to the volume integral of the triple product of the Fourier functions in the bands comprising ℓ, ℓ', and ℓ'':

$$C(\ell,\ell',\ell'') \simeq \int dx \int_{k\simeq\ell/\ell_b} d^3k \int_{k'\simeq\ell'/\ell_b} d^3k' \int_{k''\simeq\ell''/\ell_b} d^3k'' \cdot \{ \tag{24}$$
$$e^{(i(\underline{k}+\underline{k}'+\underline{k}'')\underline{x})} |\underline{v}(\underline{k},t)|\,|\underline{v}(\underline{k}',t)|\,|\underline{v}(\underline{k}'',t)|\}$$

where $\underline{v}(\underline{k},t)$ is the Fourier transform of $\underline{v}(\underline{x},t)$. A triad of Fourier modes with wavenumbers $\underline{k}$, $\underline{k}'$, and $\underline{k}''$ contributes to the integral if and only if $(k + k' + k'') = 0$. Certain correlations are kinematically required to be zero. For example, if $\ell \gg \ell''$ and $\ell' \gg \ell''$ then $C(\ell,\ell',\ell'') = 0$. $C(\ell,\ell',\ell'')$ also reflects the temporal correlation among $v(\ell)$, $v(\ell')$, and $v(\ell'')$. If motions among disparate scales have no phase coherence, then $C(\ell,\ell',\ell'')$ is small. If $C(\ell,\ell',\ell'')$ is insignificant except when $\ell \simeq \ell' \simeq \ell''$, then the rate of numerical dissipation from $v(\ell)$ is approximately

$$\frac{1}{2}\left(\frac{\Delta}{\ell}\right)\frac{V(\ell)^3}{\ell} \tag{25}$$

Equation (25) shows that if $v(\ell)^3/\ell$ is independent of ℓ (Kolmogorov theory) or if $v(\ell)^3/\ell^2$ increases with decreasing ℓ, then the small scales are preferentially dissipated. On the other hand, if $v(\ell)$ decreases exponentially with ℓ (cf. the dissipative spectrum in figure 1), then equation (25) shows that the numerical viscosity is most effective at ℓ_b not Δ. If $C(\ell,\Delta,\Delta)$ is large and if $v(\ell)/\ell$ increases with decreasing ℓ, then the rate of numerical dissipation is proportional to

$$\frac{1}{2}\,V(\ell)V(\Delta)^2/\Delta \tag{26}$$

If $C(\ell,\ell,\ell_b)$ is large and if $v(\ell)/\ell^2$ decreases with decreasing ℓ, then the rate is proportional to

$$\frac{1}{2}\left(\frac{\Delta}{\ell_b}\right) v(\ell)^2 v(\ell_b)/\ell_b \qquad (27)$$

The rates in equations (26) and (27) are most effective at ℓ_b, not Δ, and flows with correlations and spectra that produce these rates are not well simulated with a numerical viscosity of this type. Even in the best case where equation (25) is applicable, there is significant dissipation at ℓ_b. In particular, the Reynolds number based on $v(\ell_b)$, ℓ_b, and the numerical viscosity at ℓ_b implied by equation (25), $1/2\,\Delta v(\ell_b)$, is

$$R = 2\,\ell_b/\Delta \qquad (28)$$

For a 3-dimensional calculation with 10^6 grid points, the flow has an effective Reynolds number of only 200, which corresponds to a viscous laminar flow. Therefore, this numerical viscosity is not useful for simulating turbulence.

More sophisticated techniques are needed to dissipate selectively the kinetic energy from the small motions without disturbing the rest of the large-scale flow. For example, Siggia and Patterson (1978) in a pseudo-spectral simulation of turbulence enhanced the viscosity in the modal equations of motion that govern the smallest 15% of the flow and left the equations for the large modes unchanged. The magnitude of their enhancement was adjusted until the energy spectrum obeyed Kolmogorov's $v(\ell) \propto \ell^{1/3}$ scaling law. This technique is useful in spectral calculation but cannot be implemented in an easy way with finite-differences. Furthermore, in complicated flows one does not know a priori the shape of the energy spectrum, so some other method must be found for determining the correct amount of viscous enhancement.

Smagorinsky (1963) has modeled the subgrid-scale forcing, $\underline{Q}$, as an eddy-viscosity in a way that can be easily implemented in finite-difference calculations. The eddy-viscosity is more selective than the numerical viscosity in removing the energy from the small scales while leaving the large scales untouched. Of course, it is not as selective as the modal method of Siggia. To see how an eddy-viscosity works, consider a flow with constant density. The subgrid forcing, $\underline{Q}$, can be written as the divergence, $\nabla \cdot \underline{\underline{R}}$, where $\underline{\underline{R}}$ is the symmetric tensor

$$R_{jk} = \overline{v'_j v'_k} + \overline{\bar{v}_j v'_k} + \overline{v'_j \bar{v}_k} \qquad (29)$$

where we have assumed that the viscous terms are negligible and

that the derivative operator commutes with the large-scale filter operator. It will be useful to break $\underline{\underline{R}}$ into a trace and a traceless part

$$\tau_{jk} \equiv R_{jk} - R_{mm}\,\delta_{jk}/3 \tag{30}$$

$$\tilde{P}/\rho \equiv R_{mm}/3 \tag{31}$$

$$\underline{Q} = \nabla \cdot \underline{\underline{\tau}} + \nabla\tilde{P}/\rho \tag{32}$$

The filtered Navier-Stokes equation is

$$\frac{\partial \underline{\bar{v}}}{\partial t} = -\overline{(\underline{\bar{v}}\cdot\nabla)\underline{\bar{v}}} - \frac{1}{\rho}\nabla(\bar{P}+\tilde{P}) - \nabla\cdot\underline{\underline{\tau}} + \nu\nabla^2\underline{\bar{v}} \tag{33}$$

Smagorinsky proposed treating $\nabla\cdot\underline{\underline{\tau}}$ like a viscous term by letting $\underline{\underline{\tau}}$ be proportional to the large scale strain, $\underline{\underline{\bar{S}}}$,

$$\underline{\underline{\tau}} = -2\nu_e\,\underline{\underline{\bar{S}}} \tag{34}$$

where

$$\bar{S}_{jk} = \frac{1}{2}\left(\frac{\partial \bar{v}_j}{\partial x_k} + \frac{\partial \bar{v}_k}{\partial x_j}\right) \tag{35}$$

and ν_e is the eddy-viscosity. Note that we could not let $\underline{\underline{R}}$ itself be proportional to $\underline{\underline{S}}$ because $\underline{\underline{S}}$ is traceless (as is $\underline{\underline{\tau}}$) and $\underline{\underline{R}}$ is not. The turbulent pressure-head, $\tilde{P}$, need not be determined; the quantity $(\bar{P}+\tilde{P})$ is found in the usual manner by requiring that $\underline{v}$ be divergence-free. Taking the divergence of equation (33) we obtain

$$\frac{1}{\rho}\nabla^2(\bar{P}+P) = -\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\left(\bar{v}_j\bar{v}_k + \tau_{jk}\right) \tag{36}$$

Smagorinsky proposed an eddy-viscosity of

$$\nu_e = c^2\Delta^2(2\bar{S}_{jk}\bar{S}_{jk})^{1/2} \tag{37}$$

where c is a constant of order unity. His choice of ν_e was motivated by the following argument. We have already shown that ν_e should be proportional to $v(\Delta)\Delta$

$$\nu_e = c'\,v(\Delta)\Delta \tag{38}$$

where c' is a constant of order unity. Although the value of Δ is known, it is not easy to determine $v(\Delta)$ in a finite-difference calculation. The value of $v(\Delta)$ can be estimated from an energy balance. The rate at which energy is drained from the large-scale

flow in equation (33) is

$$\nu_e \bar{S}_{jk}\bar{S}_{jk} \tag{39}$$

From the Kolmogorov theory of homogeneous, isotropic turbulence, the rate at which energy enters the subgrid-scales is

$$v(\Delta)^3/\Delta \tag{40}$$

Equating expressions (39) and (40), we find

$$v(\Delta) = (c')^{1/2}\,\Delta\,(\bar{S}_{jk}\bar{S}_{jk})^{1/2} \tag{41}$$

which combined with equation (38) gives the Smagorinsky eddy-viscosity in equation (37). The weak point of Smagorinsky's argument is that if the flow does not behave like homogeneous, isotropic turbulence, then equation (40) is incorrect. We show in a later section how equation (40) can be modified for inhomogeneous flow.

Smagorinsky's isotropic eddy-viscosity is superior to the previously discussed numerical viscosity. If the flow at disparate length-scales is poorly correlated, then the dissipation rate at ℓ is

$$c^2\,(\frac{\Delta}{\ell})^2\,\frac{v(\ell)^3}{\ell} \tag{42}$$

If $v(\ell)^3/\ell$ is independent of ℓ (Kolmogorov theory) or if $v(\ell)^3/\ell^2$ increases with decreasing ℓ, then most of the energy is drained from the smallest resolvable scales. The effective Reynolds number at ℓ_b made from $v(\ell_b)$, ℓ_b, and the viscosity at ℓ_b implied by equation (42), $c^2\Delta^2 v(\ell_b)/\ell_b$ is

$$R = \frac{1}{c^2}\,(\frac{\ell_b}{\Delta})^2 \tag{43}$$

This Reynolds number is greater than the one computed with the numerical viscosity (equation 28) by a factor of (ℓ_b/Δ). For a 3-dimensional calculation with 10^6 grid points and with c=0.1 (see below) the effective large-scale Reynolds number is 10^6. This Reynolds number is characteristic of a nearly inviscid, turbulent flow, so the Smagorinsky eddy-viscosity may be useful in computing turbulent astrophysical flows.

We note that both Smagorinsky's eddy-viscosity and the numerical viscosity are positive-definite sinks of energy, but for any particular length-scale, ℓ, the viscous term can be a source of energy. However, if $(\bar{S}_{jk}\bar{S}_{jk})$ and Δ are both independent of position in the fluid, then ν_e is a sink of energy at all

length-scales. We also note that if the flows at all resolvable length-scales are well-correlated, then it is possible for the Smagorinskii eddy-viscosity to be most dissipative at ℓ_b not Δ .

Eddy-viscosities have been used extensively in meteorological calculations (cf. Smagorinsky , Manake, and Holloway, 1965). Lilly (1967) used Smagorinsky ´s eddy-viscosity to compute homogeneous, isotropic turbulence and found that c=0.17 in equation (37) gave the best scaling law for the velocity. Deardorff (1970) used an eddy-viscosity to compute an inhomogeneous, anisotropic plane channel flow (plane Poiseuille flow) and found good agreement with the laboratory values of the statistical measures of turbulence by setting c=0.1. With c=0.17 he found the flow too dissipative. Not surprisingly, he also discovered that in regions where the flow is strongly anisotropic or inhomogeneous, the agreement between his calculations and experiments was poor.

ANALYTIC CALCULATION OF SUBGRID-SCALE FLOW

Treating the large-scale effects of the subgrid scales as an eddy-viscosity is clearly inadequate at any place in the fluid or at any scale in the spectrum where the net downward cascade of energy is not approximately equal to $v(\ell)^3/\ell$, or where the cascade is not stationary in time or is not isotropic. But even in stationary, homogeneous, isotropic turbulence the eddy-viscosity may be inaccurate. We need a mathematical framework in which to develop a better model of $\underline{Q}$. To understand how the subgrid-scales affect the large scales, it is first necessary to understand how the large-scale motions create the subgrid-scale flow. Kolmogorov's picture of turbulence is based on two major assumptions. One is that small-scale turbulent flow is universal (i.e., the small-scale flow is independent of the detailed structure of the large-scale flow and is therefore the same in every turbulent flow). Strongly related to the idea of universality is the concept of locality, whereby the only scales, ℓ' , that directly affect $v(\ell)$ are those with $\ell' \simeq \ell$. Non-locality is kinematically allowed whenever there is a nonlinear interaction among three or more Fourier modes whose wavevectors add to zero. However, non-locality also requires that there be a strong correlation among disparate length-scales. Universality and locality have never been proven analytically or experimentally, but if it can be shown that universality exists among turbulent flows with very different large-scale coherent structures, then locality is proven.

The second assumption by Kolmogorov is that the energy rate, ε , is the only information that passes from the large-scale

to the small-scale flow. Each length-scale has energy cascading downward through it at the same rate, so the subgrid flow has scale-similarity (i.e., apart from a scaling factor, the turbulent velocity at ℓ is indistinguishable from the turbulent velocity at any other small length). Scale-similarity has been shown experimentally to be incorrect (Van Atta and Park, 1972); the intermittent behavior of turbulence at length-scale, ℓ , was measured to be a function of both ε and ℓ/ℓ_b . Intermittency is a familiar sight to anyone who has watched a ripple, or cat's-paw, sweep across the surface of a lake on a gusty day. In laboratory flows, turbulence is accompanied by spatial and temporal bursts that make the flow appear spotty or streaky. For example, in channel flow with Reynolds numbers of several hundred, turbulent spots appear near the walls, and most of the transfer of energy from the large-scale flow to the small-scales is confined to these irregular regions. In turbulent flows, the velocity is a function of time but by ensemble or time-averaging (denoted by double angle brackets) the distribution function of $v(\ell)$ about its mean value can be measured. The velocity and its derivatives have large departures from their mean values for longer amounts of time in an intermittent flow than they do in a non-intermittent flow. The flatness factor of the velocity distribution is defined to be $\langle\langle v(\ell)^4 \rangle\rangle / (\langle\langle v(\ell)^2 \rangle\rangle)^2$ and is one measure of the likelihood of large departures from the mean; hence, the flatness measures the intermittency of the flow. Experiments show that the flatness factor increases with $(\ell_b - \ell)/\ell_b$, which means that the velocity at ℓ is not only a function of ε but also has a memory of the length of the cascade from ℓ_b to ℓ. Heuristically, it appears that the longer the energy cascades, the more likely it is to produce a large deviation from the mean.

Besides ε and ℓ/ℓ_b , it is not known what other quantities associated with the large-scale flow are needed to determine the small-scale flow. It has been conjectured (Lorenz, 1969) that all of the details of the large-scale flow are needed to compute the small-scale flow, and vice versa; if any minor change occurs in the small-scales the large-scale flow changes dramatically (e.g., a butterfly flapping its wings determines tomorrow's weather). If the "butterfly effect" is important, then numerical simulation and modeling of turbulent astrophysical flows is impossible. Even if the mean values of a large-scale turbulent flow are amenable to subgrid-scale modeling, if the large-scale intermittency depends upon detailed knowledge of the small scales, many astrophysical calculations are impossible. For example, consider the calculation of stellar convection. If we are interested in computing the extent of mixing due to the convective overshoot, then knowledge of the mean convective flow may not be useful. If intermittent bursts carry the convective penetration hundreds of times farther than the mean overshoot and if the intermittency occurs on a

time-scale much longer than an eddy-turnover time (but much shorter than a stellar lifetime) then it is the intermittency, not the mean convective overshoot, that is relevant to stellar mixing. Throughout the remainder of this paper, by necessity, we shall assume that the large-scale quantities of interest can be computed with subgrid-scale models.

There are several analytic theories of turbulence based on the ideas of universality, localness, and scale-similarity that attempt to calculate the subgrid-scale forcing. To provide a theoretical basis for the previous phenomenological discussion of $\underline{Q}$, we now sketch briefly the analytic determination of the subgrid-scale forcing. Almost all theories of incompressible turbulence begin with the Fourier transform of the Navier-Stokes equation.

$$(\frac{\partial}{\partial t} + \nu k^2) v_\alpha(\underline{k},t) = -\frac{i}{2} P_{\alpha B\gamma}(\underline{k}) \sum_{\underline{P}} v_\beta(\underline{P},t) V(\underline{k}-\underline{P},t) \qquad (44)$$

where the tensor $P_{\alpha\beta\gamma}(\underline{k})$ is defined

$$P_{\alpha\beta\gamma}(\underline{k}) \equiv k_\beta P_{\alpha\beta}(\underline{k}) + k_\gamma P_{\alpha\beta}(\underline{k}) \qquad (45)$$

with

$$P_{\alpha\beta}(\underline{k}) \equiv \delta_{\alpha\beta} - k_\alpha k_\beta / |k|^2 \qquad (46)$$

The nonlinear term in equation (44) is the Fourier transform of the divergence-free component of $-(\underline{v}\cdot\nabla)\underline{v}$ (thereby including the effects of the pressure term). Exact evaluation of the nonlinear term is impossible. The type of approximation used is what distinguishes each theory of turbulence. Multiplying equation (44) by $v(k,t)$ one obtains an equation for the energy $E(k,t)$. In principle, all of the well-known theories are capable of casting the energy equation into the general form

$$\{\frac{\partial}{\partial t} + k^2[\nu + \nu_e(k,t)]\} E(k,t) = k^4 A(k,t) \qquad (47)$$

where the nonlinearities are absorbed into an eddy-viscosity, $\nu_e(k,t)$ and a forcing term, $A(k,t)$, where ν_e and A depend only weakly on k. The ν_e term in equation (47) is readily understood, at least on a phenomenological level, since we have already shown that the effect of the small-scales on the large-scales (via the nonlinear terms) is to act like a viscosity. The forcing term that is proportional to k^4 is heuristically understood by observing that in the limit of stationary, inviscid turbulence there will be an equi-partition of energy among all the modes in Fourier space (Rose and Sulem, 1978). Since the number of Fourier modes with

wavenumber k is proportional to $4\pi k^2$, the energy spectrum of a stationary, inviscid flow is

$$E(k) \propto k^2 \qquad (48)$$

The k^4 forcing term is needed in equation (47) to be consistent with equation (48). A(k,t) is created by the non-local nonlinear interaction of Fourier modes. It is the ability of A(k,t) and $\nu_e(k,t)$ to selectively drain and force energy out of or into selective wavevectors that permits the small-scale flow to control the large-scale motions in a way that can be qualitatively different from the way ordinary viscosity controls laminar flow. The Red Spot of Jupiter, stellar dynamos, and super-granulation are probably all due to the non-viscous-like properties of the small-scale forcing.

One method of approximating the nonlinear terms in equation (44) is by using the renormalization group (Forster, Nelson, and Stephen, 1977), which uses the assumption of scale-similarity with propagator techniques from field theory. Intermittency cannot be treated correctly because of the self-similar assumption. The more widely used method of solving equation (44) is Kraichnan's direct interaction theory (1959) and its offspring: Lagrangian history direct interaction, (Kraichnan, 1965), the test field model (Kraichnan,1971, Newman and Herring, 1979), and eddy-damped quasi-normal Markovian theory (Orszag, 1970, 1974). One proceeds by approximating cumulants or moments of the velocity. Equation (44) expresses the time rate of change of the velocity as a function of second-order moments of the velocity. Multiplying equation (44) by $\underline{v}(\underline{k}')$ yields an equation for the second-order moments in terms of the third-order moments. By continuing to multiply equation (44) by the velocity, a hierarchy of moment equations is built. Approximations are made when the hierarchy is truncated by either discarding the highest moments or approximating them as functions of lower order moments.

With respect to numerical modeling of $\underline{Q}$, the most usable results of analytic turbulence theory are the evaluation of A(k,t), $\nu_e(k,t)$, and (when the heat equation is also used) $k_e(t)$, the eddy-thermal-diffusivity. Closed form expressions for these quantities as integrals of the energy and thermal variance spectra weighted by functions of the turbulent time-scales have been obtained by Herring(1973), Basdevant, et al. (1978), Leslie and Quarini (1978), Chollet and Lesieur(1981), and others. The predictions are in reasonable agreement with measurements of decay times in wind tunnel grid-turbulence and with experimental measurements of ν_e/k_e, the eddy Prandtl number, in a turbulent boundary-layer (Fulachier and Dumas, 1976). Unfortunately, most of the

analytic theories of turbulence include arbitrary constants whose values must be determined by comparison with experiment. Furthermore, none of the theories are easily generalizable to flows with inhomogeneous or anisotropic motions of the large-scale. No one has found a method of numerically simulating the large-scale flow with a subgrid-scale forcing based on an analytic theory of turbulence that is more useful than one of the phenomenological treatments of $\underline{Q}$. We conclude this paper with four examples of calculations where an eddy-viscosity model of $\underline{Q}$ is inadequate. We improve the model of $\underline{Q}$ in a heuristic manner while bearing in mind the theoretical implications of this section.

IMPROVED MODELS OF THE SUBGRID FLOW

Computation of the constant in the eddy-viscosity

In a simulation of laboratory flow, the value of the constant, c', which appears in equation (38) for the eddy-viscosity, is determined by adjusting its value until the numerical calculation agrees with the laboratory data. In simulations of astrophysical flow, we do not have this luxury. We require that the numerical method itself determine the value of c'. Such a method is of special importance in stellar convection where the subgrid flow diffuses entropy as well as momentum. The values of the constants that appear in the eddy-viscosity and in the eddy entropy diffusivity will not be the same, and their ratio determines the eddy-Prandtl number. One method for determining the constants uses scale-similarity (Marcus, 1980). This technique, like the others discussed previously, divides the velocity field into a resolvable component $\bar{\underline{v}}$ with length-scales greater than Δ and a subgrid component $\underline{v}'$. However, it also introduces an intermediate length-scale, Δ', with $\ell_b > \Delta' > \Delta$. The length, Δ', is small enough so that indigent numericists who can only afford to use the coarser resolution, Δ', in their simulation will still produce accurate results if they use an eddy-viscocity. For simplicity, we consider a flow with periodicity ℓ_b and use a spectral method where $\bar{\underline{v}}$ is defined to contain all Fourier modes $\underline{k}$ with $2\pi/\Delta > |\underline{k}| \geq 2\pi/\ell_b$ and where $\underline{v}'$ contains modes with $2\pi/\ell_d \geq |\underline{k}| \geq 2\pi/\Delta$. Define $\underline{v}''$ to be the part of the velocity field that is resolvable with the fine resolution, Δ, but unresolvable with the coarse resolution, i.e., $\underline{v}''$ contains modes $2\pi/\Delta > |\underline{k}| \geq 2\pi/\Delta'$. The large-scale filtering function used in the fine resolution calculation, indicated by an overbar, discards all modes with $|\underline{k}| \geq 2\pi/\Delta$. The filtering function in the coarse calculation, indicated by a double overbar, discards modes with $|\underline{k}| \geq 2\pi/\Delta'$. In a constant density fluid, the coarse resolution calculation solves the equation

$$\partial\bar{\bar{\underline{v}}}/\partial t = \overline{\overline{-(\bar{\bar{\underline{v}}}\cdot\nabla)\bar{\bar{\underline{v}}} - \nabla(\bar{\bar{P}}+\tilde{\tilde{P}})/\rho + \nu\nabla^2\bar{\bar{\underline{v}}}}} + \overline{\overline{\nabla\cdot[c'\Delta' v(\Delta')\nabla\bar{\bar{\underline{v}}}]}} \tag{49}$$

and the fine resolution solves

$$\partial\bar{\underline{v}}/\partial t = -\overline{[(\bar{\bar{\underline{v}}}+\underline{v}'')\cdot\nabla](\bar{\bar{\underline{v}}}+\underline{v}'')} - \nabla(\bar{P}+\tilde{P})/\rho + \nu\nabla^2(\bar{\bar{\underline{v}}}+\underline{v}'') + \overline{\nabla\cdot[c'\Delta v(\Delta)\nabla(\bar{\bar{\underline{v}}}+\underline{v}'')]} \tag{50}$$

where $v(\Delta)$ and $v(\Delta')$ are the characteristic velocities at Δ and Δ', respectively, and where $\tilde{P}$ and $\tilde{\tilde{P}}$ are determined so that the right sides of equations (49) and (50) are divergence-free. Note that

$$\bar{\underline{v}} = \bar{\bar{\underline{v}}} + \underline{v}'' \tag{51}$$

Since both calculations are assumed accurate, the computed value of $\bar{\bar{\underline{v}}}$ is the same in both calculations, and since scale-similarity is assumed, the values of c' in equations (49) and (50) are the same. By multiplying equation (49) by $\bar{\bar{\underline{v}}}$ and integrating over the volume, we obtain the rate of change of energy in $\bar{\bar{\underline{v}}}$

$$\partial\bar{\bar{E}}/\partial t = -\int d^3x\ 2\nu\bar{\bar{S}}_{jk}\bar{\bar{S}}_{jk} + 2c'\Delta' v(\Delta')\bar{\bar{S}}_{jk}\bar{\bar{S}}_{jk} \tag{52}$$

while equation (50) predicts

$$\partial\bar{\bar{E}}/\partial t = -\int d^3x\ 2\nu\bar{\bar{S}}_{jk}\bar{\bar{S}}_{jk} + 2c'\Delta v(\Delta)\bar{S}_{jk}\bar{\bar{S}}_{jk} - \int d^3x\ \bar{\bar{\underline{v}}}\cdot\left[(\bar{\underline{v}}\cdot\nabla)\underline{v}''\right] \tag{53}$$

The correct value of c' is determined at each time step by requiring that the two expressions for $\bar{\bar{E}}(t)$ are equal

$$c'(t) = \frac{\int d^3x\ \bar{\bar{\underline{v}}}\cdot\left[(\bar{\underline{v}}\cdot\nabla)\underline{v}''\right]}{2\int d^3x[\Delta' v(\Delta')\bar{\bar{S}}_{jk} - \Delta v(\Delta)\bar{S}_{jk}]\bar{\bar{S}}_{jk}} \tag{54}$$

The numerator in equation (54) is the rate at which energy flows from $\bar{\bar{\underline{v}}}$ into $\underline{v}''$, i.e., from motion with length-scales greater than Δ' into scales ℓ'' where $\Delta' \geq \ell'' > \Delta$. In almost all physically

realistic calculations, this rate is positive. In a spectral simulation of Boussinesq convection, Marcus demonstrated that this method of determining c' is stable and rapidly convergent. The reason for rapid convergence is shown by the following argument. If $\Delta' v(\Delta')$ and $\Delta v(\Delta)$ are independent of position, then

$$\int d^3x \, \Delta v(\Delta) \, \bar{\bar{S}}_{jk} \bar{\bar{S}}_{jk} = \Delta v(\Delta) \int d^3x \, \bar{\bar{S}}_{jk} \bar{\bar{S}}_{jk} \tag{55}$$

and

$$c'(t) = \frac{\frac{1}{2}\int d^3x \, \bar{\bar{v}} \cdot \left[(\bar{v} \cdot \nabla) v'' \right] / \int d^3x \, \bar{\bar{S}}_{jk} \bar{\bar{S}}_{jk}}{\Delta' v(\Delta') - \Delta v(\Delta)} \tag{56}$$

The quantity, $\bar{\bar{S}}_{jk} \bar{\bar{S}}_{jk}$, is positive, and $[\Delta' v(\Delta') - \Delta v(\Delta)]$ can be assumed positive, since in any realistic flow $v(\ell)$ decreases with decreasing ℓ. If the value of c' is too small, the flow responds by piling up energy in the smallest scale Δ, making $v(\Delta)$ large and $[\Delta' v(\Delta') - \Delta v(\Delta)]$ small. The small denominator in equation (56) then makes c' increase at the next time step. Similarly, if the value of c' is too large, the energy is preferentially dissipated from the smallest mode, the denominator in equation (56) becomes large, and the value of c' is reduced. We have never found the pile up of energy at Δ so severe that the denominator changes sign and produces a negative eddy-viscosity. By requiring that the rate of change of the entropy variance be the same in calculations with fine and coarse resolution, an equation analogous to equation (54) can be derived for the constant in the eddy entropy diffusivity. Marcus used this technique to determine numerically the eddy-Prandtl number of convection in water.

Inhomogeneous flow

The above method can be used in inhomogeneous flows provided that Δ, Δ' $v(\Delta)$, $v(\Delta')$, and c' are treated as functions of position. For example, $v(\Delta, \underline{x})$ is the characteristic velocity at $\underline{x}$, with length-scale Δ, averaged over a volume of fluid of size λ where $\ell_b >> \lambda >> \Delta$. $c'(\underline{x})$ is determined by requiring that the rates of change of energy in the coarse and fine resolution calculations are the same at each position in the fluid. This method fails in inhomogeneous flows where the physical mechanism by which the subgrid-scale flow removes energy from the large-scale flow is a function of position. For example, consider a numerical simulation of large-scale turbulent solar convection. Equation (38) for the eddy-viscosity and equation (39) for the energy loss rate from the large-scale flow are valid. However, equation (40) for the dissipation rate is not correct because the subgrid-scale effects at the center and at the base of the convective zone are very different. Near the center, the numerically

resolvable components of the velocity that are smaller than a density scale-height behave like incompressible flow when they lose their kinetic energy to the subgrid-scales. In this region, equation (40) is adequate. At the base of the convective zone, the downward velocity runs into a stably stratified layer and causes kinetic energy to change into potential and thermal energy via small-scale density and pressure fluctuations, respectively. In this region, the subgrid-scales do not act like an incompressible flow and, the dissipation in equation (40) must be replaced with one that models the effects of a subgrid-scale compressible flow.

We illustrate the use of a spatially dependent form of the eddy-viscosity with a simpler inhomogeneous flow for which there is an abundance of laboratory data. In channel flow, the velocity breaks up into small, highly dissipative vortices near the boundaries. In these regions, the subgrid-scale dissipation rate depends directly on the viscosity. Following the work of Moin, et al. (1978), in this boundary region we replace equation (40) for the dissipation rate with

$$\nu v(\Delta)^2/\Delta^2 \tag{57}$$

Equation (57) combined with equations (38) and (39) yields

$$\nu_e = 2c'' \Delta^4 \bar{S}_{jk}\bar{S}_{jk}/\nu$$

where c" is a constant of order unity. Equation (58) is valid only in the boundary regions where it replaces Smagorinsky's eddy-viscosity (equation 37). Moin et al. define y_c as the distance from the wall where the average value of ν_e from equation (58) is equal to the average value of ν_e from equation (37). In the region between the wall and y_c, they use equation (58) and exterior to this region, they use equation (37). This improved eddy-viscosity reproduces the statistics of the turbulent flow near the wall much more faithfully than the numerical simulations that use Smagorinsky's eddy-viscosity everywhere (Deardorff, 1970).

Intermittency

None of the previously discussed subgrid-scale models allow intermittency. As we have shown, it may be necessary to model the intermittency in $\underline{Q}$ to simulate accurately the mixing properties of convection. We propose a model of the subgrid-scale flow based on work by Bell and Nelkin (1977). Bell and Nelkin calculated intermittency in homogeneous, isotropic turbulence by solving a set of heuristic equations that was designed to mimic the nonlinear cascade of energy. Although they did not develop their model with the

intention that it be used with a direct numerical simulation of the large-scale velocity, the incorporation can be done easily. In Bell and Nelkin's model, the velocity is represented by N scalars v(i,t), where v(i,t) is the characteristic velocity of a band of modes with wavelengths between $\ell_b/2^{i-1}$ and $\ell_b/2^i$. The characteristic wavelength of the i^{th} band is $\ell(i)$ and $\ell(i+1)=\ell(i)/2$. N is chosen large enough so that $\ell(N)$ is less than ℓ_d. Solar convection, with a Reynolds number of 10^{14}, can be modeled with this method because the number of variables, N, need only be greater than 35. The v(i,t) satisfy the following evolution equations

$$\frac{dv(i,t)}{dt} = a_1\{v(i-1),t)^2 - 2v(i,t)v(i+1,t) + a_2[v(i-1,t)v(i,t) - 2v(i+1,t)^2]\}/\ell(i) - \nu v(i,t)/\ell(i)^2 \quad i = 1,2,\ldots,N \tag{59}$$

where a_1 and a_2 are constants of order unity and where $v(0,t) = v(N+1,t) = 0$. These equations force the cascade to be local since v(i,t) is influenced only by itself and by v(i-1,t) and v(i+1,t). The nonlinear terms in equation (59) represent the nonlinear terms in the Navier-Stokes equation that allow the energy to cascade through the different length-scales. The constant, a_2 determines the relative importance of the downward cascade (large to small scales) to the upward cascade (small to large scales). Like the integral of the nonlinear terms in the Navier-Stokes equation, the nonlinear terms in equation (59), when summed over all i, conserve energy. The last term in equation (59) is the viscous dissipation. Bell and Nelkin found that this simple set of equations reproduce many of the statistical properties of intermittent turbulence. Marcus,et al. (1983) generalized the cascade model to inhomogeneous, compressible convection for use in astrophysical calculations.

Incorporating the Bell-Nelkin cascade model in a numerical simulation is easiest in a spectral or pseudo-spectral computation. Motions with length-scales greater than Δ are governed by the Navier-Stokes equation and directly simulated. Motions smaller than 2Δ are treated with the cascade model (equation 59), where the i^{th} band is now defined to contain length-scales between $\Delta/2^{i-2}$ and $\Delta/2^{i-1}$. The velocity with scales between Δ and 2Δ is represented both in the direct large-scale simulation and in the cascade model. Equation (59) for v(i,t) with i=1 is discarded and replaced with

$$v(1,t) = [\int d^3x\ \underline{v}_\Delta(\underline{x},t)\cdot\underline{v}_\Delta(\underline{x},t)/\int d^3x]^{1/2} \tag{60}$$

where

$$\underline{v}_{\Delta}(\underline{x},t) \equiv \int_{\pi/\Delta \leq |\underline{k}| < 2\pi/\Delta} d^3k\ \bar{\underline{v}}(\underline{k},t) \qquad (61)$$

and where $\bar{\underline{v}}(\underline{k},t)$ is the Fourier transform of the large-scale velocity. The Navier-Stokes equation for the large-scale velocity is solved as usual except that an eddy-viscosity term, $\nabla \cdot \nu_e \nabla \underline{v}$ is used with the modes with wavelengths between Δ and 2Δ. The eddy-viscosity is determined by conservation of energy. The rate at which kinetic energy enters the subgrid-scales from the large scales is determined by multiplying equation (59) by v(i,t) and summing the product from i=2 to i=N.

$$a_1 v(1,t)v(2,t)[v(1,t)+a_2 v(2,t)]/\ell(2) \qquad (62)$$

The rate at which the eddy-viscosity removes energy from the large-scale flow is

$$\nu_e \int d^3x\ (\nabla \underline{v}_{\Delta}) \cdot (\nabla \underline{v}_{\Delta}) \qquad (63)$$

Equating these two rates we obtain

$$\nu_e = \frac{a_1 v(1,t)v(2,t)[v(1,t)+a_2 v(2,t)]}{\ell(2) \int d^3x\ (\nabla \underline{v}_{\Delta}) \cdot (\nabla \underline{v}_{\Delta})} \qquad (64)$$

This method of computing ν_e, allows intermittency in the subgrid-scales and has been implemented by the author. However, at the present time, no comparisons have been made between experimental values of the intermittency (flatness factor, etc.) and the numerical simulations.

Eddy-viscosity in rotating flows

Our last example of the need for a more sophisticated eddy-viscosity is rotating flow. Not only the magnitude, but also the functional form of the model of the subgrid-scale forcing must be correct. In particular, it is the detailed form of $\underline{Q}$ that determines the way in which angular momentum is expelled in a contracting proto-star, permits a dynamo to form in the sun, and is responsible for the longevity of the Red Spot of Jupiter. $\underline{Q}$ is usually chosen to be the sum of a gradient and the divergence of a traceless tensor, $\underline{\underline{\tau}}$, (see equations 30 - 32) so that mass, momentum, and angular momentum are conserved. A bad choice of $\underline{\underline{\tau}}$ will ruin a calculation. For example, consider a computation of a collapsing proto-star that uses Smagorinsky's $\underline{\underline{\tau}}$ which is defined in equations (34) and (37). Changing the value of c changes the relative rates of mass infall and angular momentum redistribution,

but the stable equilibrium rotation curve of a newborn star will remain the same. The form, not the magnitude, of $\underline{\underline{T}}$ determines whether a star rotates as a solid body, has constant angular momentum per unit mass ($v_\phi \propto r^{-1}$), or has a more exotic rotation law. Without knowing the correct form of $\underline{\underline{T}}$, it is pointless to undertake numerical calculations where the results depend strongly on the angular momentum distribution.

Modeling $\underline{Q}$ as the divergence of a tensor implies that the subgrid-scales affect the large-scale flow by diffusing some physical quantity. Therefore, the correct choice of $\underline{\underline{T}}$ depends on the transport properties of the small scale flow. Smagorinsky's eddy-viscosity diffuses momentum, but momentum does not always appear to be the correct quantity to diffuse. To see what the small-scale flow transports, let us for the moment consider a simpler eddy diffusivity. In compressible flows, the entropy is an adiabatic invariant - i.e., in the limit of no dissipation, the covariant derivative of the entropy is zero.

$$(\frac{\partial}{\partial t} + \underline{v}\cdot\nabla)s = 0 \qquad \text{when} \quad \nu = k = 0 \tag{65}$$

The small-scale turbulent flow carries entropy fluctuations with it. If the time-averaged turbulence is homogeneous, the flow becomes isentropic. In compressible fluids, the temperature is not an adiabatic invariant, so the flow does not become isothermal. Since the large-scale effect of the subgrid-flow is the diffusion of entropy, not temperature, an eddy entropy diffusivity must be included in the equations of motion, and it would be incorrect to include an eddy thermal diffusivity.

Even in incompressible flow, momentum is not an exact adiabatic invariant because of the pressure. However, in rotating cylindrical flows, there is experimental evidence that the angular momentum per unit mass (with respect to the axis of rotation) acts like an adiabatic invariant (cf. DiPrima and Swinney, 1981). In rotating spherical flows, experiments indicate that this is not true (Wimmer, 1976), and it has been suggested that in rapidly rotating spherical flows, the effective adiabatic invariant is the enstrophy (i.e., the square of the vorticity). It can be easily shown that for axisymmetric flows, the angular momentum per unit mass is an exact adiabatic invariant, and for 2-dimensional flows (no velocity component parallel to the axis of rotation) the enstrophy is an exact adiabatic invariant. At the present time, there is no way of analytically or observationally determining what the correct invariant (if one exists!) is in a rotating star. However, numerical simulations (with no eddy-viscosity) of cylindrical Couette flow show clearly that the small, but numerically resolvable, scales advect angular momentum as an adiabatic

invariant (Marcus, 1983); a high resolution numerical simulation might reveal what quantity behaves like an adiabatic invariant in a rotating compressible sphere of fluid and thereby allow us to choose $\underline{\underline{T}}$ properly in a stellar calculation.

DISCUSSION

We have shown that in many astrophysical flows it is impossible to simulate all of the scales of interest. Ignoring the small scales sometimes causes a numerical instability. Their neglect does not always produce an unstable code which is unfortunate because we are often lulled into thinking that if a calculation does not blow up, it produces an accurate solution. We have given some examples of spurious results produced in underresolved computations, and we warn the reader to be wary of them. In homogeneous, isotropic flows without intermittent behavior, a simple eddy-viscosity is probably sufficient to model the large-scale effects of the subgrid-scale flow. However, in most flows of astrophysical interest, the subgrid-scale effects are due to complicated physical processes that are a function of position and time. Analytic theories of turbulence can provide us with some guidance in modeling the small scales, but, at the present time, phenomenological models based on the physics of the subgrid-scale flow are necessary.

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REFERENCES

Basdevant, C., Lesieur, M. & Sadourny, R. 1978 J. Atmos. Sci. 35, 1028.

Bell, T. L. & Nelkin, M. 1977 Phys. Fluids 20, 345.

Chollet, J. P. & Lesieur, M. 1981 J. Atmos. Sci. 38, 2747.

Deardorff, J. W. 1970 J. Fluid Mech. 41, 453.

Deardorff, J. W. & Willis, G. E. 1967 J. Fluid Mech. 28, 657.

DiPrima, R. C. & Swinney, H. L. 1981 in Hydro. Instabilities and the Transition to Turbulence, Topics in Applied Physics, Vol. 45 (ed. H. L. Swinney and J.P. Gollub) p. 139 Springer.

Forster, D., Nelson, D. R., & Stephen, M. J. 1977 Phys. Rev. A 16, 732.

Fulachier, L. & Dumas, R. 1976 J. Fluid Mech. 77, 257.

Herring, J.R. 1973 in Proc. Langly Working Conf. on Free Turbulent Shear Flows. NASA SP321 (1973) Langley Research Center, Langley, VA.(1973).

Kraichnan, R. H. 1959 J. Fluid Mech. 5, 497.

Kraichnan, R. H. 1965 Phys. Fluids 8, 575.

Kraichnan, R. H. 1971 J. Fluid Mech. 47, 513.

Kolmogorov, A. N. 1941 C. R. Acad. Sci. U. R. S. S. 30, 299.

Leslie, D. C. & Quarini, G.L. 1978 J. Fluid Mech. 91 65.

Lilly, D. K. 1967 in Proc. of the IBM Scient. Comp. Symp. on Env. Science. IBM Form No. 320-1951.

Lorenz, E. N. 1969 Tellus 21, 289.

Marcus, P. S. 1980 Ap. J. 240, 203.

Marcus, P. S. 1981 J. Fluid Mech. 103, 241.

Marcus, P. S. 1983 submitted to J. Fluid. Mech.

Marcus, P. S., Press, W. H. & Teukolsky, S. A. 1983 Ap. J. 267, 795.

Moin, P., Reynolds, W. C., & Ferziger, J. R. 1978 NASA Technical Report Number TF-12 NASA Ames.

Newman, G. R. & Herring, J. R. 1979 J. Fluid Mech. 94, 163.

Orszag, S. A. 1970 J. Fluid Mech. 41 363

Orszag, S. A. 1974 in Proc. 1973 Les Houches Summer School (ed. R. Balian & J.-L. Peabe), pp. 237-374 Gordon & Breach.

Rose H. & Sulem, P. L. 1978 J. Phys. (Paris) 39, 441.

Siggia, E. D. & Patterson, G. S. 1978 J. Fluid. Mech. 86, 567.

Smagorinsky, J. 1963 Monthly Weather Rev. 91, 99.

Smagorinsky, J., Manake, S. & Holloway, J. 1965 Monthly Weather Rev. 93, 727.

Toomre, J., Gough, D.O., & Spiegel, E.A. 1977 J. Fluid Mech. 79, 1

Toomre, J., Zahn, J.-P., Latour, J. & Spiegel, E. A. 1976 Ap. J. 207, 545.

Van Atta, C. W. & Park, J. 1972 in Statistical Models and Turbulence (ed. M. Rosenblatt and C. Van Atta) p. 402 Springer.

Wimmer, M. 1976 J. Fluid Mech. 78, 317.